Irrationality of degenerations of Fano fibrations

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Structure of the talk

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

1. Setup and degenerations of del Pezzo surfaces.
2. Bounding irrationality in higher-dimensional klt Fano varieties.
3. Elements of toroidal geometry.
4. Sketchy proof of the main result.

This is a joint work with Prof. C. Birkar.

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Fano varieties

Definition

A **Fano variety** is a projective variety X with mild singularities and ample $-K_X$.

Example

- projective space $\mathbb{P}^n$.
- smooth complete intersections $X = H_1 \cap \cdots \cap H_r \subset \mathbb{P}^n$ with $\sum_{i=1}^r \deg H_i \leq n$.
- weighted projective spaces $\mathbb{P}(a_0, \ldots, a_n)$ (which has quotient singularities).
- $\mathbb{P}^1$ is the only Fano variety of dimension 1.
- Fano varieties of dimension 2 are called **del Pezzo surfaces**.

Definition

A **Fano fibration** is a contraction $X \to Z$ where X has mild singularities and $-K_X$ is ample over Z. A **contraction** $f: X \to Z$ is a projective morphism of varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Z$; in particular, f is surjective with connected fibres.

Singularities of pairs

Definition

A **pair** (X, B) consists of a normal quasi-projective variety X and an $\mathbb{R}$-divisor B with coefficients in $[0, 1]$ such that $K_X + B$ is $\mathbb{R}$-Cartier. In this case, we say that B is a **boundary**.

Definition

Let $\phi\colon W \to X$ be a log resolution of a pair (X, B), and write

$$K_W + B_W = \phi^*(K_X + B).$$

Let $\mu_D B_W$ be the coefficient of B_W at a prime divisor D on W. Then the **log discrepancy** of D with respect to (X, B) is defined as

$$a(D, X, B) := 1 - \mu_D B_W.$$

We say (X, B) is **lc** (resp. **klt**, resp. ϵ-**lc**) if $a(D, X, B)$ is ≥ 0 (resp. > 0, resp. $\geq \epsilon$) for every divisor D on an arbitrary log resolution $W \to X$.

Degenerations of del Pezzo surfaces

Let $f: X \to Z$ be a contraction, H a general fibre, and F a special fibre over a closed point $z \in Z$. Then we say that F is a **degeneration** of H.

Example

A degeneration of $\mathbb{P}^1$ is a chain of rational curves. For example, degeneration of a smooth conic curve can be a union of two lines.

Definition

Let X be a variety of dimension n over $\mathbb{K}$.

- X is rational if there is a birational map $\mathbb{P}^n \dashrightarrow X$,
- X is unirational if there is a dominant rational map $\mathbb{P}^n \dashrightarrow X$,
- X is rationally chain connected (RCC) if for a pair of very general closed points $x_1, x_2 \in X$ there is a connected curve $C \subset X$ containing x_1, x_2 such that every irreducible component of C is rational, and
- X is rationally connected (RC) if we can take C irreducible.

Degenerations of del Pezzo surfaces

Theorem [Campana92, Kollár-Miyaoka-Mori92, Zhang04, . . .]

A klt Fano variety X is rationally connected (RC).

Remark

The result fails if X is only lc, e.g., cone over an elliptic curve.

Definition

Let X be a variety of dimension n. We say that X is **ruled** (resp. **uniruled**) if there is a variety Y of dimension $n-1$ and a map

$$\mathbb{P}^1 \times Y \dashrightarrow X$$

which is birational (resp. dominant).

Remark

Rational $\Longrightarrow$ unirational $\Longrightarrow$ RC $\Longrightarrow$ RCC $\Longrightarrow$ uniruled.

Degenerations of del Pezzo surfaces

Theorem [Matsusaka68] [Kollár96, IV.1.6]

Let $f\colon X \to Z$ be a contraction to a smooth curve Z with X normal and irreducible. If the generic fibre of f is (geometrically) ruled, then a special fibre of f over a closed point $z \in Z$ also has ruled components.

Theorem [Kollár96, IV.3.5.3]

Degenerations of rationally chain connected (RCC) varieties are RCC.
Degenerations of rationally connected (RC) varieties are RCC.

Example ("Rational" degenerates to "non-rational")

A smooth del Pezzo surface (rational) can degenerate to a non-rational singular del Pezzo surface. For example, pick an elliptic curve $E \subset \mathbb{P}^2$ which gives a cone $F \subset \mathbb{P}^3$. Pick another smooth del Pezzo surface $H \subset \mathbb{P}^3$, then we can connect F and H by a pencil of surfaces. The singular surface F is **non-rational**; however, F is ruled, that is, $F \dashrightarrow C \times \mathbb{P}^1$ for some smooth curve C. This happens because F has lc singularities but not klt singularities.

Degenerations of del Pezzo surfaces

Let C be a smooth curve. We can measure the "non-rationality" of C by its **genus** $g(C)$ and **gonality** $\mathrm{gon}(C)$.

Definition

By **gonality** $\mathrm{gon}(C)$ of a smooth projective curve C, we mean the smallest natural number r such that there is a finite morphism $C \to \mathbb{P}^1$ of degree r.

Theorem [Birkar-Loginov (2021), Theorem 1.1]

Fix a positive real number t. Assume that $f : X \to Z$ is a klt Fano fibration where $\dim X = 3$ and $\dim Z = 1$. Assume that F is an irreducible fibre and that (X, tF_{red}) is lc. Then,

1. F_{red} is birational to $\mathbb{P}^1 \times C$, where C is a smooth projective curve with gonality $\mathrm{gon}(C)$ bounded depending only on t,
2. if $t > \frac{1}{2}$, then the genus $g(C)$ is bounded, and
3. if $t = 1$, then the genus $g(C) \leq 1$.

Degenerations of del Pezzo surfaces

Remark

• To bound the genus $g(C)$, it is necessary to assume that $t > \frac{1}{2}$.
• A counter-example is given in [Birkar-Loginov (2021), Example 2.3] where $t = \frac{1}{2}$ and the gonality $\text{gon}(C)$ is bounded, but $g(C)$ can be arbitrarily large.
• The gonality is not bounded neither if (X, tF_{red}) is not lc.

Theorem [Birkar-Loginov (2021), Theorem 1.3]

Fix a positive real number t . Assume that $(X, B) \to Z$ is a Fano-type log Calabi-Yau fibration where $\dim X = 3$ and $\dim Z \geq 1$. Assume that D is a component of B with coefficient $\geq t$ contracted to a point on Z. Then:

1. D is birational to $\mathbb{P}^1 \times C$, where C is a smooth projective curve with gonality $\text{gon}(C)$ bounded depending only on t,
2. if $t > \frac{1}{2}$, then the genus $g(C)$ is bounded, and
3. if $t = 1$, then the genus $g(C) \leq 1$.

Degenerations of del Pezzo surfaces

Definition

A variety X over Z is called **of Fano type over** Z if there is a boundary Γ such that (X, Γ) is a klt and $-(K_X + \Gamma)$ is ample$/Z$.

Theorem [Hacon-McKernan07, Corollary 1.5]

Assume that X is of Fano type over Z. Then, every fibre of $X \to Z$ is RCC.

Definition

A **log Calabi-Yau fibration** $(X, B) \to Z$ is an lc pair (X, B) over Z such that the underlying morphism $X \to Z$ is a contraction and

$$K_X + B \sim_{\mathbb{R}} 0/Z.$$

If in addition X is of Fano type over Z, then we say that (X, B) is a **Fano-type log Calabi-Yau fibration**.

Degenerations of del Pezzo surfaces

- The Fano-type condition can be removed in dimension 2.
- The absolute case ($\dim Z = 0$) is allowed in dimension 2.

Theorem [Birkar-Loginov (2021), Theorem 1.5]

Let u be a positive real number. Let $(S, \Delta) \to Z$ be a log Calabi-Yau fibration ($K_X + \Delta \sim_{\mathbb{R}} 0/Z$) where S is of dimension 2. Assume that the coefficient of a component D of Δ is $\geq u$ and D is contracted to a point on Z. Then:

1. the gonality of D is bounded depending only on u,
2. if $u > \frac{1}{2}$, then the genus $g(D^\nu)$ of the normalisation D^ν of D is bounded depending only on u,
3. if $u = 1$, then $g(D^\nu) \leq 1$, and
4. if $\dim Z \geq 1$, then $g(D^\nu) \leq 1$.

Remark

The absolute case and non-Fano situation in dimension 2 are applied to prove the case when $\dim X = 3$ and $\dim Z \geq 1 \implies$ difficulties when $\dim X \geq 4$.

Bounding irrationality in higher dimensions

What about higher-dimensional Fano fibrations over curves?

Definition (analogue of "gonality" in higher dimensions)

Given an irreducible projective variety X of dimension n, we define the **degree of irrationality** of X as

$$\mathrm{irr}(X) = \min \left\{ \delta > 0 \;\middle|\; \begin{array}{c} \exists \text{ degree } \delta \text{ dominant rational map} \\ X \dashrightarrow \mathbb{P}^n \end{array} \right\}.$$

For convenience, we also say that the **irrationality of** X is $\mathrm{irr}(X)$.

Question [Birkar-Loginov (2021), Question 1.4]

Fix a positive real number $t > 0$ and natural number d. Suppose that $f: X \to Z$ is a **klt Fano fibration** over a smooth curve Z, where $\dim X = d$. Assume that D is the reduction of an irreducible fibre of f such that (X, tD) is lc. Is it true that there is a rational map $D \dashrightarrow C$, where the general fibres are rationally connected and C is a smooth projective variety with bounded degree of irrationality?

Bounding irrationality in higher dimensions

Theorem [Birkar-Q (2024)]

Fix positive real numbers $\epsilon > 0$, $t \in (0, 1]$ and a natural number d. Assume that $f : X \to Z$ is a klt Fano fibration with $\dim X = d$ such that

1. Z is a smooth curve,
2. X is ϵ-lc over the generic point of Z, and
3. F is the reduction of an irreducible fibre of f such that (X, tF) is lc.

Then, there exists a natural number r depending only on ϵ, t, d such that there is a dominant rational map $F \dashrightarrow C$ whose general fibres are **rational** and C is a smooth projective variety with degree of irrationality $\leq r$.

Bounded and birationally bounded families of varieties

The condition on ϵ-lc implies boundedness.

Definition

(1). A **couple** (X, D) consists of a quasi-projective variety X and a reduced Weil divisor D on X.

(2). Let $\mathcal{Q}$ be a set of couples (X, D). We say that $\mathcal{Q}$ is **bounded** if

- there exist finitely many projective morphisms $V_i \to T_i$ of varieties and reduced divisors C_i on V_i,
- for each $(X, D) \in \mathcal{Q}$, there exist an i and a closed point $t \in T_i$ such that

$$\begin{array}{ccccc} (V_i, C_i) & \longleftarrow & (V_{i,t}, C_{i,t}) & \xrightarrow{\phi} & (X, D) \\ \downarrow & & \downarrow & & \\ T_i & \longleftarrow & t & & \end{array}$$

  where ϕ is an isomorphism, and $V_{i,t}$ and $C_{i,t}$ are the fibres over t of the morphisms $V_i \to T_i$ and $C_i \to T_i$ respectively.

Boundedness of ϵ-lc Fano varieties

Definition

If $D = 0$ for all $(X, D) \in \mathcal{Q}$, we say the family $\mathcal{Q}$ of varieties X is **bounded**. Moreover, in this case, we say that the family of varieties $\mathcal{Q}$ is **birationally bounded** if in the definition above we take ϕ as a birational map.

Theorem [Alexeev94]

For any fixed $\epsilon > 0$, ϵ-lc del Pezzo surfaces are bounded.

Theorem [Kollár-Miyaoka-Mori92]

Smooth Fano varieties of any given dimension form a bounded family.

Theorem [Birkar16, Borisov-Alexeev-Borisov conjecture]

Let d be a natural number and $\epsilon > 0$ a positive real number. Then, ϵ-lc Fano varieties of dimension d form a bounded family.

Elements of toroidal geometry (toric varieties)

Definition

A **toric variety** X of dimension d is an irreducible variety containing an algebraic torus $\mathbb{T}_X \simeq \mathbb{G}_m^d$ as a Zariski open subset such that the action of $\mathbb{T}_X$ on itself extends to an algebraic action of $\mathbb{T}_X$ on X.

Remark

If D is the **toric boundary** of X, that is, D is the reduction of the complement of the torus $\mathbb{T}_X$ of X, it is well-known that (X, D) is lc and $K_X + D \sim 0$. In this case, we say (X, D) is a **toric couple**.

Example

Let X be the affine space $\mathbb{A}^d$, and D is the union of d coordinate hyperplanes. Then, (X, D) is a toric couple.

Remark

Toric varieties are rational.

Elements of toroidal geometry (toroidal couples)

Definition

Let (X, D) be a couple. We say the couple is **toroidal** at a closed point $x \in X$ if there exist a **normal affine toric variety** W and a closed point $w \in W$ such that there is a $\mathbb{K}$-algebra isomorphism

$$\widehat{\mathcal{O}}_{X,x} \to \widehat{\mathcal{O}}_{W,w}$$

of completions of local rings so that the ideal of D is mapped to the ideal of the torus-invariant divisor $C \subset W$, that is, the complement of the big torus $\mathbb{T}_W$ of W. We call $\{(W, C), w\}$ a **local toric model** of $\{(X, D), x\}$.

Proposition [Artin69, Corollary 2.6]

There is a common étale neighbourhood of (X, x) and (W, w).

Definition

We say the couple (X, D) is **toroidal** if it is toroidal at every closed point.

Elements of toroidal geometry (toroidal embeddings)

Remark

In literature, the open immersion $U_X := X \setminus \mathrm{Supp} D \subset X$ is called a **toroidal embedding**. We usually use the notions toroidal couples and toroidal embeddings interchangeably. Moreover, if the embedding $U_X \subset X$ (or equivalently, the couple (X, D)) is clear from the context, we could just say that X is a **toroidal variety** or X **has a toroidal structure**.

Example

Let (X, D) be a log smooth couple. Then, (X, D) is toroidal.

Remark

Toroidal varieties are not necessarily rational. Any function field of a variety can be realised as the function field of a toroidal variety by taking log resolutions.

Elements of toroidal geometry (properties)

Lemma

Let (X, D) be a toroidal couple. Then X is normal and Cohen-Macaulay, $K_X + D$ is Cartier, and (X, D) is an lc pair.

Proposition

Let (X, D) be a toroidal couple, and let F be a divisor over X. Then, the log discrepancy $a(F, X, D)$ is a non-negative integer. In particular, if $a(F, X, D) < 1$, we must have $a(F, X, D) = 0$ and $F \to \mathrm{centre}_X F$ has rational general fibres.

Proof.

Pass to common étale neighbourhoods and extract divisors. □

Elements of toroidal geometry (toroidal morphisms)

Definition

Let (X, D) and (Y, E) be couples and let $f\colon X \to Y$ be a morphism of couples. Let $x \in X$ be a closed point and $y = f(x)$. We say $(X, D) \to (Y, E)$ is a **toroidal morphism at** x if there exist local toric models $\{(W, C), w\}$ and $\{(V, B), v\}$ of $\{(X, D), x\}$ and $\{(Y, E), y\}$ respectively, and a toric morphism $W \to V$ of affine normal toric varieties so that we have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,x} & \longrightarrow & \widehat{\mathcal{O}}_{W,w} \\ \uparrow & & \uparrow \\ \widehat{\mathcal{O}}_{Y,y} & \longrightarrow & \widehat{\mathcal{O}}_{V,v} \end{array}$$

where the vertical maps are induced by the given morphisms and the horizontal maps are isomorphisms induced by the local toric models. We say the above morphism is **toroidal** if it is toroidal at every closed point of X. Equivalently, we call the corresponding morphism $f\colon (U_X \subset X) \to (U_Y \subset Y)$ a **toroidal morphism of toroidal embeddings**.

Sketch proof of dimension 3 (setup)

Fix positive real numbers $\epsilon > 0$, $t \in (0, 1]$. Assume that $f : X \to Z$ is a klt Fano fibration with $\dim X = 3$ such that

1. Z is a smooth curve,
2. X is ϵ-lc over the generic point of Z, and
3. F is the reduction of an irreducible fibre of f such that (X, tF) is lc.

Goal

The goal is to show that F admits a structure $\pi : F \dashrightarrow C$ where C is a smooth variety with bounded irrationality and a general fibre of π is rational.

Remark (boundedness of general fibres)

The general fibres of $X \to Z$ are ϵ-lc irreducible del Pezzo surfaces, hence they form a bounded family of varieties by [Alexeev94] or B-BAB.

Sketch proof of dimension 3 (bir. to toroidal morphisms)

There is a commutative diagram:

$$\begin{array}{ccccc}
(U_{Y'} \subset Y') & \xrightarrow{m_X} & Y & \overset{\phi}{\dashleftarrow} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
(U_{Z'} \subset Z') & \xrightarrow[m_Z]{} & Z & = & Z
\end{array}$$

such that

1. ϕ is a birational map,
2. every fibre of $g\colon Y \to Z$ is bounded (relatively bounded),
3. ϕ can be chosen so that it does not contract any curve over η_Z,
4. f' is a toroidal morphism of toroidal embeddings,
5. m_X and m_Z are projective birational morphisms, and
6. the general fibres of f' are bounded ([Abramovich-Temkin-Włodarczyk20]).

Sketch proof of dimension 3 (lc centre of toroidal couples)

Definition

Let (X, B) be a pair where B is a boundary, and let $X \to Z$ be a contraction. Let $T = \lfloor B \rfloor$ and $\Delta = B - T$, and $n \in \mathbb{N}$ a natural number. An *n*-complement of $K_X + B$ over a point $z \in Z$ is of the form $K_X + B^+$ such that over some neighborhood of z we have the following properties:

- (X, B^+) is lc,
- $n(K_X + B^+) \sim 0$, and
- $nB^+ \geq nT + \lfloor (n+1)\Delta \rfloor$.

By the existence of complements ([Birkar19, Theorem 1.8]), there exists an $n \in \mathbb{N}$ depending only on d, t such that there exists an n-complement $K_X + B^+$ of $K_X + tF$ over Z, that is, there exists a boundary B^+ on X such that

- (X, B^+) is log canonical,
- $n(K_X + B^+) \sim 0/Z$,
- $tF \leq B^+$, and
- $a(F, X, B^+) < 1$.

Sketch proof of dimension 3 (lc centre of toroidal couples)

$$\begin{array}{ccccc} (U_{Y'} \subset Y') & \xrightarrow{m_X} & Y & \overset{\phi}{\dashleftarrow} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ (U_{Z'} \subset Z') & \xrightarrow[m_Z]{} & Z & = & Z \end{array}$$

Write

$$K_{Y'} + B' = (\phi^{-1} \circ m_X)^*(K_X + B^+).$$

Proposition

Denote by $D' := Y' \setminus U_{Y'}$ the toroidal boundary. Then, the toroidal modifications m_X and m_Z can be chosen in the way such that $B' \leq D'$.

Corollary

Denote by C' the centre of F on Y'. Then, we have $a(F, Y', D') < 1$, that is, C' is an lc centre of (Y', D').

$$
\begin{array}{ccccc}
(U_{Y'} \subset Y') & \xrightarrow{m_X} & Y & \overset{\phi}{\dashleftarrow} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
(U_{Z'} \subset Z') & \xrightarrow[m_Z]{} & Z & = & Z
\end{array}
$$

Proposition

The toroidal modifications f', m_X, and m_Z can be chosen in the way such that the support of the fibre of f' over $z \in Z \simeq Z'$ is contained in D'.

Sketch proof of dimension 3 (boundedness of slc pairs)

• Denote by S the reduction of the fibre of f' over $z \in Z \simeq Z'$. Then, C' is an lc centre of (Y', D') contained in S. By adjunction, we can write

$$K_S + D_S = (K_{Y'} + D')|_S$$

for a boundary divisor D_S. If (S, D_S) is a semi-log canonical (slc) pair, then C' is an lc centre of the two-dimensional slc pair (S, D_S).

• Taking a sufficiently ample$/Z'$ divisor on Y' etc., then we are in the situation to apply the main theorem of [Hacon-M^cKernan-Xu14] on the boundedness of slc pair, which shows that (S, D_S) is bounded.

• Therefore, C' is also bounded as it is an lc centre of a bounded set of slc pairs (hence C' has bounded gonality and arithmetic genus if C' is an irreducible curve).

Sketch proof of dimension 3 (bounded gonality and genus)

• Let C be a resolution of C', then there is a rational map $\pi\colon F \dashrightarrow C$. As C' is an lc centre of the toroidal couple (Y', D'), a general fibre of π is rational.

• The dimension of C gives several possibilities on the structure of F:

1. if $\dim C = 0$, then F is a rational surface,
2. if $\dim C = 1$, then C has bounded gonality $\mathrm{gon}(C)$ and genus $g(C)$, a general fibre of π is isomorphic to $\mathbb{P}^1$, hence F is birational to $\mathbb{P}^1 \times C$, and
3. if $\dim C = 2$, then F is birationally bounded as it is birational to C'.

Recall that F must be ruled, hence F is bir. to $\mathbb{P}^1 \times E$ for some smooth curve E.

1. We can take $E = \mathbb{P}^1$.
2. E is isomorphic to C which has bounded gonality and genus.
3. As F is birationally bounded in this case, F has bounded degree of irrationality, hence E also has bounded gonality which is bounded from above by the degree of irrationality of F (compare the "stable-irrationality" and gonality). The irregularity ($= \dim \mathrm{H}^1(\Sigma, \mathcal{O}_\Sigma)$) of the surface $\mathbb{P}^1 \times E$ is equal to the genus $g(E)$. Then there are only finitely many possible values for the irregularity of $\mathbb{P}^1 \times E$, so $g(E)$ is also bounded from above.

Sketch proof of dimension 3 (some remarks)

Remark

We have shown that F is birational to $\mathbb{P}^1 \times E$ where E has both gonality and genus bounded. However, if we drop the condition on ϵ-lc of general fibres (boundedness condition), then [Birkar-Loginov, Example 2.3] shows that the gonality $\mathrm{gon}(E)$ is always bounded, but $g(E)$ could be arbitrarily large.

Remark

Our approach via toroidal geometry works in any dimension.

$$\begin{array}{ccccc} (U_{Y'} \subset Y') & \xrightarrow{m_X} & Y & \overset{\phi}{\dashleftarrow} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ (U_{Z'} \subset Z') & \xrightarrow[m_Z]{} & Z & = & Z \end{array}$$

For any $d = \dim X$, $C' := \mathrm{centre}_{Y'} F$ is birationally bounded.

Thank you!